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# Engineering Notes

## Traveling-Salesman Problem for a Class of Carrier–Vehicle Systems

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### I. Introduction

THE complexity of many scenarios envisioned for future autonomous systems, ranging from planetary exploration to rescue missions, requires a broad range of capabilities for individual units (often including air, ground or sea mobility and sophisticated multimodal sensor suites and actuation devices), which cannot be implemented with a single platform class. Rather, it may be necessary to coordinate several specialized units to attain complex objectives in a reliable, timely, and efficient way [1]. While considerable progress has been made on cooperative control of networks of homogeneous vehicles (see, e.g., [2–5]), heterogeneous networks are still relatively poorly understood. In such a direction, recent efforts (see, e.g., [6,7]) have been undertaken for spreading the adoption of unmanned systems in real-world operational scenarios. In particular, see [8], where the employment of cooperating mobile robots, often denoted as multiple mobile robot systems, is a clear example of the capabilities achievable by combining the characteristics of heterogeneous vehicles with complementary features. To optimally exploit the different capabilities of each individual unit in obtaining the desired final behavior the team is required to be suitably coordinated via advanced planning and control algorithms.

This Note concentrates on systems of heterogeneous vehicles, arising from the combination of 1) a slow autonomous carrier vehicle (typically a ship) with a long operating endurance and 2) a faster vehicle (typically an aircraft) with a limited operative endurance. The carrier is able to transport the faster vehicle, as well as to deploy, recover, and service it. Even though this two-vehicle system is very simple it has relevant applications, and many path planning and coordination problems of interest, similar to those introduced in [9–11] for other systems may be defined for it. In the preliminary work [12] the determination of the optimal trajectories connecting up to two given points has been detailed. This Note extends previous results to the case in which $n$ points have to be visited and the visiting sequence is not given a priori.

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### II. Carrier–Vehicle System

The system we are going to deal with is composed of two different vehicles (see Fig. 1): a *carrier*, whose variables and functions will be denoted by the subscript $c$, and a *vehicle*, denoted by the subscript $v$. In the following we will refer to the combined system as the *carrier–vehicle system*. The vehicles will be considered as material points belonging to the Euclidean space $\mathbb{R}^2$ and their position will be denoted, respectively, as $p_c(t) = [x_c(t) y_c(t)]^T$ and $p_v(t) = [x_v(t) y_v(t)]^T$. It is assumed that the carrier has a single integrator behavior, and thus it is able to follow any continuous path with a speed that is limited to be lower than or equal to a certain maximal velocity $V_c > 0$, i.e., $\|\dot{p}_c(t)\| := \sqrt{\dot{x}_c(t)^2 + \dot{y}_c(t)^2} \in [0, V_c]$ for all $t \geq 0$. Regarding the carried vehicle, we need to distinguish between two different situations:

1) When it is not carried, it can follow any continuous path with a speed lower than or equal to $V_v > V_c$, i.e.,

$$\|\dot{p}_v(t)\| := \sqrt{\dot{x}_v(t)^2 + \dot{y}_v(t)^2} \in [0, V_v]$$

2) When it is carried, its position coincides with that of the carrier, viz., $p_v(t) = p_c(t)$.

To model the limited operating range of the faster vehicle, it is assumed that it can leave the carrier position, i.e., $p_v \neq p_c$, for no more time than $\bar{a}$ units of time. Moreover, anytime the vehicle comes back to the carrier, i.e., $p_v \equiv p_c$, its remaining operating endurance is instantaneously restored to $\bar{a}$.

### III. Ordered Visit of *n* Points

Among the large variety of planning problems that can be formulated considering the above presented carrier–vehicle system, this Note will focus on a special scenario that was inspired, in particular, by rescue operations (see, for instance, [7]), where the faster vehicle is required to land back to the carrier after having visited each target point (e.g., to transport victims on the main vehicle as soon as possible). Within this framework the first mission planning problems we can define is the following:

*Problem 1.* Let an initial point $p_0$, a desired terminal point $p_f$ and a list of $n$ points $q_{\text{list}} = [q_1, \ldots, q_n]$ be given with $p_c(0) = p_v(0) = p_0$. Determine the minimum-time trajectory allowing each point to be visited by the carried vehicle in an ordered way by respecting, for each point $q_i$, a given sequence of takeoff, visiting the new point-landing prescriptions and ending, for both vehicles, at the terminal point $p_f$.

By exploiting the fact that, in the absence of constraints, the optimal trajectory between two points is a straight line covered at the maximal speeds for both the carrier and the carried vehicle, the above problem may be formulated as the following optimization problem:

$$\begin{aligned}
&\min_{p_{to,i}, p_{l,i}, t_{p_{l,i}, p_{to,i+1}}, t_{p_{to,i}, p_{l,i}}} \sum_{i=0}^{n} t_{p_{l,i}, p_{to,i+1}} + \sum_{i=1}^{n} t_{p_{to,i}, p_{l,i}} \quad t_{p_{to,i}, p_{l,i}} \leq \bar{a} \\
&i = 1, \ldots, n \quad \frac{1}{V_c} \|p_{l,i} - p_{to,i}\| \leq t_{p_{to,i}, p_{l,i}}, \quad i = 1, \ldots, n \\
&\frac{1}{V_v} (\|q_i - p_{to,i}\| + \|p_{l,i} - q_i\|) \leq t_{p_{to,i}, p_{l,i}}, \quad i = 1, \ldots, n \\
&\frac{1}{V_c} \|p_{l,i} - p_{to,i+1}\| \leq t_{p_{l,i}, p_{to,i+1}}, \quad i = 0, \ldots, n
\end{aligned} \tag{1}$$

where the unknowns to be determined are the collection of points $p_{to,i}$, $p_{l,i} \in \mathbb{R}^2$, $i \in \{1, \ldots, n\}$, representing the takeoff and the landing points that allow the vehicle to optimally visit the $i$th target, and the time intervals $t_{p_{l,i}, p_{to,i+1}} \in \mathbb{R}$, $t_{p_{to,i}, p_{l,i}}$, $i \in \{1, \ldots, n\}$

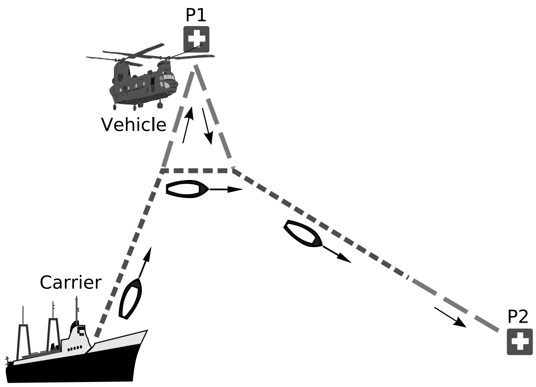


**Fig. 1 Carrier–vehicle system on a rescue scenario.**

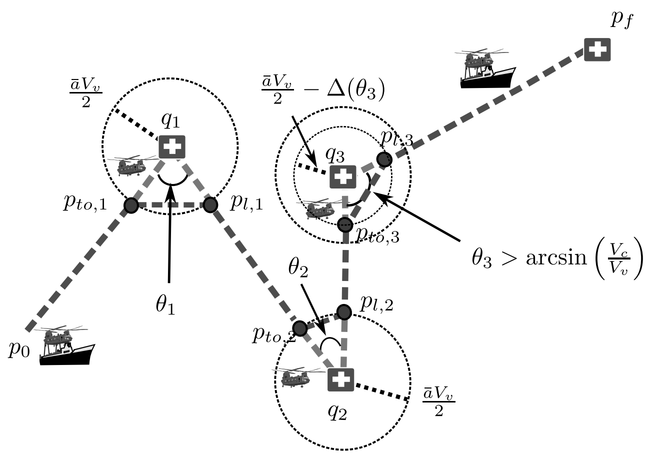


**Fig. 2 Geometric interpretation behind the upper bound $t_U(\ell, n, \theta_{\mathrm{list}})$.**

representing, respectively, the time that the carrier will employ to reach the $(i+1)$th takeoff point from the $i$th landing point and the time that the vehicle will spend between the takeoff and the subsequent landing instants needed to visit the $i$th point.

For simplicity, $p_{l,0} = p_0$ and $p_{to,n+1} = p_f$ will be assumed. Note that since the objective function is linear and the constraints are convex, Eq. (1) is a convex optimization problem and then the optimum may be efficiently computed through numerical solvers.

Although an exact analytic solution is unknown (beside a few special cases like those discussed in [12]), it is possible to analytically determine upper-bounds and lower bounds to the optimal solution of Problem 1. Namely, it is possible to prove that a lower bound to the optimal cost of Problem 1 is always given by

$$t_L(\ell, n) = \max\{(\ell/V_c - nV_v\bar{a}/V_c + n\bar{a}), \ell/V_v\} \tag{2}$$

where $\ell$ is the length of the shortest path that a single holonomic vehicle would undertake to complete the visit of all points of interest, i.e.,

$$\ell = \sum_{i=1}^{n+1} d_{i-1,i}$$

where $d_{0,1} := \|p_0 - q_1\|$, $d_{i-1,i} := \|q_{i-1} - q_i\|$, $i \in \{2, 3, \ldots n\}$, $d_{n,n+1} := \|q_n - p_f\|$.

To derive an upper bound, it is enough to analytically find a feasible, not necessarily optimal, solution to Problem 1. To this end, denote $d_{\min} := \min_{i=1,\ldots,n} d_{i-1,i}$ and let $\bar{a}' = \min\{d_{\min}/V_v, \bar{a}\}$. The following upper bound

$$t_U(\ell, n, \theta_{\mathrm{list}}) = t_L(\ell, n) + \sum_{i=1}^{n} \frac{\Delta(\theta_i, \bar{a}, \bar{a}')}{V_c} \tag{3}$$

may be obtained with

$$\Delta(\theta, \bar{a}, \bar{a}') := \begin{cases} (\bar{a} - \bar{a}')V_v & \text{if } \theta \le 2\arcsin\left(\frac{V_c}{V_v}\right) \\ \bar{a}V_v - \bar{a}'V_c/\sin(\theta/2) & \text{else} \end{cases} \tag{4}$$

where $\theta_{\mathrm{list}} := [\theta_1, \theta_2, \ldots \theta_n]$, $\theta_i \in [0, \pi]$, $i = 1, \ldots, n-1$, denotes the list of the $n$ (smallest) angles formed by the segments that connect two consecutive points to be visited (for a graphical intuition, also see Fig. 2). Please note that if $\bar{a}' = \bar{a}$ and $\theta_i \le 2\arcsin(V_c/V_v)$ for all $\theta_i \in \theta_{\mathrm{list}}$, the proposed upper bound precisely matches the lower bound (2) and it is indeed the optimal cost.

## IV. Traveling-Salesman Problem

In this Note we are interested in dealing with the following carrier–vehicle traveling-salesman problem (CV-TSP):

*Problem 2 (CV-TSP).* Let $p_0$ be the same initial position for both vehicles, viz., $p_c(0) = p_v(0) = p_0$, and assume an unordered set of $n$ points $q_{\mathrm{set}} = \{q_1, \ldots, q_n\}$ to be visited be given. Determine the minimum-time trajectory allowing each point to be visited by the carried vehicle by respecting, for each point $q_i$ to be visited, a given sequence of takeoff, visiting the new point-landing prescriptions and ending, for both vehicles, to the initial point $p_0$.

As is well known, traveling-salesman problems are typical NP-hard (nondeterministic polynomial-time hard) optimization problems. For such a reason, in most practical cases, heuristics have to be used to solve them. In this Note, in order to deal with the particular TSP problem at hands, we propose a heuristic algorithm based on the Euclidean TSP (E-TSP). E-TSP is a particular case of the general TSP problem in which, given $n$ points in the space, the goal is to determine the optimal sequence that minimizes the sum of the Euclidean distances between consecutive points. One of the main feature of this class of TSP problems is that, although still NP-hard, they admit polynomial-in-time optimization schemes (see [13]). This means that for any scalar $e > 0$, it is possible to find in a polynomial time a tour whose length is, at most, $(1 + 1/e)$ times larger than the optimal length. Then, in practice, for any instance of E-TSP we can obtain an *almost*-optimal solution in a reasonable time. The CV-TSP heuristic proposed here consists of the following two steps:

1) Determine the visiting order of the almost-optimal E-TSP tour for the set of given points $\{p_0\} \cup q_{\mathrm{set}}$.

2) Use the above visiting order to solve the resulting visit of $n$ points (Problem 1) via the convex optimization formulation (1).

The idea behind this approach is that the completion time of the CV-TSP is related to the sum of the distances between points, and thus the minimization of E-TSP usually leads to achieve a reasonably good CV-TSP solution. In particular, it is possible to prove the following:

*Lemma 1.* Let the initial point $p_0$ and the set of $n$ points $q_{\mathrm{set}}$ to be visited be given. Let $\ell_{\mathrm{ETSP}}$ denote the length of the $(1 + 1/e)$-approximated optimal E-TSP tour, with $e \ge 0$, and let $\theta_{\mathrm{list}}^{\mathrm{ETSP}} := [\theta_1, \theta_2, \ldots \theta_n]$, $\theta_i \in [0, \pi]$ $(i = 1, \ldots, n-1)$ denote the list of the $n$ (smallest) angles formed by the segments connecting two consecutive points in the order given by the approximated E-TSP solution. Tthen the completion time $t_{\mathrm{CV\text{-}TSP}}^{\mathrm{heu}}$ obtained with the proposed CV-TSP heuristic has a cost that is, at most, $\varepsilon$ times greater than the optimal one, with $\varepsilon$ given by

$$\varepsilon := t_U(\ell_{\mathrm{ETSP}}, n, \theta_{\mathrm{list}}^{\mathrm{ETSP}}) \bigg/ t_L\left(\frac{\ell_{\mathrm{ETSP}}}{1 + 1/e}, n\right) \tag{5}$$

where $t_L(\cdot)$ and $t_U(\cdot)$ are defined in Eqs. (2) and (3).

*Proof.* By recalling the definition of the lower bound (2), for any given number of points $n$ the implication $\ell_1 \le \ell_2 \Rightarrow t_L(\ell_1, n) \le t_L(\ell_2, n)$ holds true. Then let $\ell$ denote the length of a generic Hamiltonian cycle for the set of points $p_0 \cup q_{\mathrm{set}}$, and let $\ell_{\mathrm{ETSP}}^{\mathrm{opt}}$ denote the length of the optimal E-TSP solution. It follows that $\ell_{\mathrm{ETSP}}^{\mathrm{opt}} \le \ell$. As a consequence, $t_L(\ell_{\mathrm{ETSP}}^{\mathrm{opt}}, n) \le t_L(\ell, n)$, which implies that $t_L(\ell_{\mathrm{ETSP}}^{\mathrm{opt}}, n)$ is a lower bound to the optimal solution of CV-TSP. Moreover, by the property of the $(1 + 1/e)$-approximated optimal

**Table 1 Comparison between the optimal solution of CV-TSP and the proposed heuristic**

| Distance | Number of cases | Optimal solution, % | Average degradation, % | Maximum degradation, % |
|---|---|---|---|---|
| Large | 500 | 88.2 | 0.028 | 1.48 |
| Normal | 1000 | 73.1 | 0.104 | 7.5 |
| Short | 500 | 52 | 0.526 | 25.1 |

**Table 2 Percentage of cases with degradation levels lower than 0.1, 1, 2.5, 5, and 10%**

| | Degradation | | | | |
|---|---|---|---|---|---|
| Distance | <0.1% | <1% | <2.5% | <5% | <10% |
| Large | 94.2% | 99% | 100% | 100% | 100% |
| Normal | 92.0% | 97.9% | 99% | 99.5% | 100% |
| Short | 82.4% | 88.6% | 93.2% | 97% | 99.4% |

E-TSP solution, it follows that $\ell_{\mathrm{ETSP}}^{\mathrm{opt}} \geq \ell_{\mathrm{ETSP}}/(1+1/e)$, where $t_L([\ell_{\mathrm{ETSP}}/(1+1/e)], n)$ is a lower bound to the optimal solution of CV-TSP as well. The proof ends by noting that $t_U(\ell_{\mathrm{ETSP}}, n, \theta_{\mathrm{list}}^{\mathrm{ETSP}})$ is an upper bound to the proposed heuristic solution. Please note that for the properties of the upper bound to Problem 1, in the particular case that the points to be visited are sufficiently far from one another (i.e., $d_{\min}/V_v > \bar{a}$) and the angles $\theta_i$ formed by the segments connecting the points in the order given by the E-TSP algorithm satisfy $\theta_i \leq 2\arcsin(V_c/V_v)$, then

$$\varepsilon = t_L(\ell_{\mathrm{ETSP}}, n) \bigg/ t_L\left(\frac{\ell_{\mathrm{ETSP}}}{1+1/e}, n\right)$$

results and the optimal sequence of points for E-TSP is also optimal for CV-TSP.

## V. Numerical Results

Numerical simulations have been undertaken to compare the optimal solution of CV-TSP with the one obtained by the proposed heuristic in a realistic scenario in which a combined unmanned surface vehicle and unmanned aerial vehicle system has to monitor some randomly generated points of interest in a certain portion of sea. The unmanned surface vehicle, which represents the carrier, is assumed to have a maximum velocity $V_c = 10$ mph, and the aircraft has a maximum flight speed $V_v = 50$ mph and a flight endurance $\bar{a} = 0.4$ h.

The targets to be visited are limited to five randomly generated points that still make the computation of the exact optimal solution in a reasonable time possible. As highlighted in Lemma 1, the distances among targets play an important role in the performance of the proposed CV-TSP heuristic. In this respect, the simulations consider three different scenarios that differ by the size of the area where the targets are generated. The first one, large distance, consists of an area of $200 \times 200$ miles, and the second and the third ones, normal distance and short distance, measure $80 \times 40$ miles and $40\times 40$ miles, respectively. The results are reported in Table 1, where the number of samples considered and the optimal solution as the percentage of cases in which the sequences generated by exactly solving the CV-TSP problem and by using the E-TSP heuristic coincide. Finally, the average and the maximal degradations in the cost of E-TSP with respect to the optimal cost of CV-TSP are shown. Note that the performance of the proposed heuristic is, at least in this case, a very tight approximation of the optimal solution. It is possible also to note that, as expected, for points generated in a larger area the heuristic and the optimal CV-TSP solutions coincide in almost all cases, whereas for points very close to each other, the average cost degradation increases. In Table 2, the statistics of the number of samples showing degradation levels, respectively, lower than 0.1, 1, 2.5, 5, and 10% are reported. As a final remark we want to point out the fact that cases in which the cost degradation is greater than 1% usually correspond to situations where several points are very close one to each other: for instance, the outlier in the short-distance case with a degradation of 25.1% corresponds to the case in which four out of the five points to be visited, as depicted in Fig. 3, are within a ball of radius 8.1 miles from the starting point.

## VI. Conclusions

This Note has addressed the traveling-salesman problem for a class of carrier–vehicle systems in which a slow carrier with infinite operating range cooperates with a faster vehicle that, on the contrary, has a limited operating range. A heuristic solution has been proposed

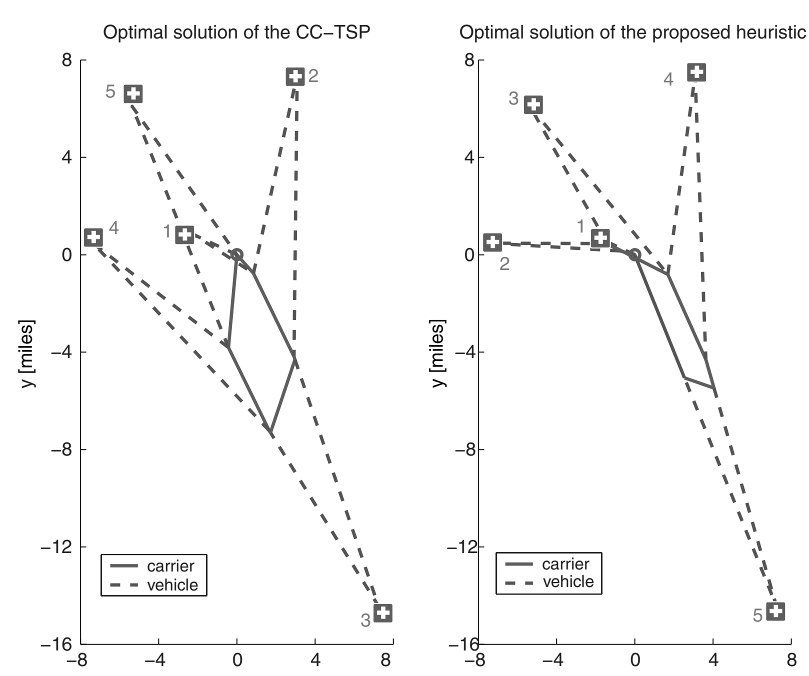


**Fig. 3 Comparisons between the optimal CV-TSP solution and the solution obtained by means of the proposed heuristic. Numbers represent the visiting order.**

and an analytical bound on its conservativeness has been derived. Extensive numerical simulations have given an insight of the effectiveness of the proposed method in some cases of interest.

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